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Abstract. In a recent paper 'On Phythian's perturbation theory for stationary homogeneous turbulence' Lee claims to show that this perturbation theory requires *a priori* knowledge of the statistical distribution of the fluid velocity and the assumption that this is gaussian. He also claims that the introduction of statistical elements into the dynamics in this theory is nonrigorous and that the derivation of the direct-interaction approximation by the truncation of the series involves the quasinormal hypothesis. These claims are disproved here and are shown to arise from a confusion of the actual fluid velocity with the gaussian velocity field on which the perturbation procedure is based.

This paper is a reply to some comments made in a recent paper 'On Phythian's perturbation theory for stationary homogeneous turbulence' (Lee 1971). His basic contention which must be challenged is that this perturbation theory necessitates *a priori* knowledge of the statistical distribution of the actual fluid velocity, and the assumption that this is gaussian, and, in addition, that the introduction of statistical elements into the dynamics is nonrigorous. Closely related to this is his statement that the derivation of the direct interaction approximation from the perturbation theory involves the quasinormal hypothesis. These points will be dealt with in turn below.

The basic equation in an abbreviated form of the usual notation is

$$\mathrm{i}\omega v = -\nu k^2 v + f + \frac{1}{VT}\sum Mvv$$

where f is the gaussian stirring force. The equation may be rewritten with the inclusion of a parameter λ as follows:

$$(\mathrm{i}\omega-\alpha)v - qf = \sum_{n=1}^{\infty}\lambda^{2n}R_{2n}v + \sum_{n=1}^{\infty}\lambda^{2n}q_{2n}f + \frac{\lambda}{VT}\sum Mvv \tag{1}$$

where the quantities α, q, R_n, q_n are nonrandom and satisfy the conditions

$$\alpha + R_2 + R_4 + \ldots = -\nu k^2$$
$$q + q_2 + q_4 + \ldots = 1. \tag{1'}$$

It is seen that the case $\lambda = 1$ corresponds to the original equation while $\lambda = 0$ gives a linear equation whose solution is

$$v_0 = \frac{1}{\mathrm{i}\omega - \alpha}\, qf.$$

The perturbation procedure consists of writing down a solution of (1) as a power series in λ the first few terms of which are seen to be

$$v = \frac{qf}{\mathrm{i}\omega-\alpha}+\frac{\lambda}{VT}\frac{1}{\mathrm{i}\omega-\alpha}\sum M\frac{qf}{\mathrm{i}\omega-\alpha}\frac{qf}{\mathrm{i}\omega-\alpha}$$
$$+\lambda^2\left(\frac{R_2 qf}{(\mathrm{i}\omega-\alpha)^2}+\frac{q_2 f}{\mathrm{i}\omega-\alpha}+\ldots\right)+\ldots.$$

As was shown in my original paper (Phythian 1969) the quantities R_n, q_n may be determined successively in terms of α and q in such a way that the power series in λ for the correlation function $\langle v\, v\rangle$ and the response function have terms in $\lambda, \lambda^2, \lambda^3, \ldots$ which are identically zero, only the zeroth order terms remaining. Inserting these expressions for $R_2, R_4 \ldots q_2, q_4, \ldots$, in (1′) then leads immediately to series for α and the correlation function $\langle v\, v\rangle$ the terms of which beyond zeroth order contain only these same quantities, that is, in the language of quantum theory we have renormalized series.

It is quite clear that this procedure involves no assumption about the probability distribution of the actual velocity field v. The series for v contains terms quadratic and of higher powers in the gaussian random function f and so v cannot itself be gaussian. Moreover, the series provides an exact formal solution of the problem. Lee's conclusion appears to be based on a confusion of v with v_0. The velocity field v_0 is gaussian but its only role is to provide a basis for the expansion procedure and it has no physical significance. It is also clear that the statistical element enters through f in the usual way.

As is well known, the direct interaction approximation is obtained by neglecting the terms $R_4, R_6 \ldots ; q_4, q_6 \ldots$ in the renormalized series mentioned above. The effect of including these terms is difficult to assess because of their complexity, but the fact that the direct interaction approximation has received some experimental support suggests that they may, in certain circumstances, be small. It is important to observe that this approximation does not involve any assumption of normality for the velocity field since the perturbation series for v still contains terms nonlinear in f. Nor does one have quasinormality as asserted by Lee. This can be seen by writing down the perturbation series for the correlation function $\langle v\, v\, v\, v\rangle$. This is found to differ from $\Sigma\langle v\, v\rangle\langle v\, v\rangle$ by a series of terms such as those having the diagram representations

etc.

These terms are quite different from those involved in the expressions for $R_4, R_6 \ldots$ and assumed to be small, so that the quasinormality condition is not even approximately satisfied.

References

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